MAS 242 Analysis II - Fall 2011 Homework 2

— Reading —

1. Read Quickly 8.1 up to p.342 (omit Sect. 8.1.1). Read 8.2 (omit proofs).

— Exercises —

2. **Harmonic functions.** Exercise cancelled (we'll do it later).

3. **A partial differential equation.** Find all differentiable functions defined on $\mathbb{R}^2$ such that

$$\frac{\partial f}{\partial x}(x,y) = xy^2, \qquad \frac{\partial f}{\partial y}(x,y) = x^2 y$$

for every point $(x,y) \in \mathbb{R}^2$.

— Problems —

4. **The derivative of a bilinear function** Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear function, i.e., $\forall y, x \mapsto B(x,y)$ is linear and $\forall x, y \mapsto B(x,y)$ is linear. Prove that B is differentiable at any point (x,y) and that $dB(x,y) \cdot (h_1,h_2) =$ ~~$B(x,h_1) + B(h_2,y)$~~ $B(h_1,y) + B(x,h_2)$.
Remark: $\Big((x_1,\dots,x_n),(y_1,\dots,y_n)\Big) \in \mathbb{R}^n \times \mathbb{R}^n$ *can be identified to* $(x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{R}^{2n}$ *so we use on* $\mathbb{R}^n \times \mathbb{R}^n$ *the Euclidean norm of* $\mathbb{R}^{2n}$*:* $\|(x,y)\| := \sqrt{x_1^2 + \cdots + x_n^2 + y_1^2 + \dots, y_n^2}$.

5. **Euler's relation.** Let f be a function of class $\mathcal{C}^1$ defined on an open set U in $\mathbb{R}^n$. We say that f is homogenous of degree k if for every $x \in U$ and every $\lambda \in \mathbb{R}$ such that $\lambda x \in U$, we have $f(\lambda x) = \lambda^k f(x)$.

   (a) Prove the Euler's relation: for every $x = (x_1,\dots,x_n) \in U$, we have
   $\sum_{i=1}^n x_i D_i f(x) =$ ~~$nf(x)$~~ $kf(x)$.
   Hint: let $g(\lambda) := f(\lambda x)$ *and show that* $g'(1)$ *equals both* $kf(x)$ *and* $df(x).x$.

   (b) We assume that $k \geq 2$. Show that, for every i, the partial derivative $D_i f$ is homogenous. What is its degree?
   Hint: let $h(x_i) := f(\lambda x)$ *and show that* $h'(x_i)$ *equals both* $\lambda^k D_i f(x)$ *and* $\lambda D_i f(x)$.

6. **The space of continuous functions on $[0,1]$ is not too big.** Show that $C[0,1]$ is separable, i.e., there exists a subset that is dense and countable.
Remark: here we use the sup-norm on $C[0,1]$.